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Published in:
Proceedings of the 33rd IEEE Conference on Decision and Control, 1994

Proceedings of the 33rd
Conference on Decision and Control
Lake Buena Vista, FL - December 1994

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Abstract

In this paper we study the singular nonlinear state feedback H_∞ problem for affine nonlinear systems. Our approach yields to a sufficient and, under an extra assumption, also necessary condition for a singular nonlinear system to construct a state feedback which leads to an L_2 gain less then a certain bound γ for the closed loop system. This condition is in terms of a set of Hamilton Jacobi Inequalities parameterised by a parameter ε.

1 Introduction

In this paper we study a singular nonlinear state feedback H_∞ problem. This kind of problems arise naturally when studying certain robustness problems such as parameter uncertainty and multiplicative uncertainty as can be seen for linear systems in [2] and [7]. We consider systems of the form

$$\begin{array}{rcl} \dot{x} & = & f(x) + g_1(x)u_1 + g_2(x)u_2 + k(x)d \\ z & = & \begin{pmatrix} h(x) \\ u_1 \end{pmatrix} \end{array} \tag{1}$$

where $u^T = (u_1^T\ u_2^T) \in \mathbb{R}^m$, $u_1 \in \mathbb{R}^{m_1}$, $u_2 \in \mathbb{R}^{m-m_1}$, $d \in \mathbb{R}^q$ and $z \in \mathbb{R}^p$. $x = (x_1, \ldots, x_n)$ are local coordinates for a smooth state space manifold M. We assume throughout that f, g_1, g_2, h, k are C^k-functions. We assume also that there exists an equilibrium $x_0 \in M$, i.e., $f(x_0) = 0$, and without loss of generality we assume $h(x_0) = 0$. After a suitable coordinate shift we may assume that $x_0 = 0$.
Our aim is to find conditions under which there exists a feedback such that the L_2-gain of the resulting closed loop system from disturbance d to output z is less than (or equal to) a certain bound γ. For the case $m_1 = m$ this is just the regular suboptimal state feedback H_∞ control problem studied in [8], [9] (see also [1] and [3]). Most of the results obtained in this paper are extensions of results about the regular nonlinear state feedback H_∞ problem obtained in these papers. For linear systems the above mentioned singular H_∞ problem has been studied quite extensively. One approach to this problem is given by Petersen, Zhou and Khargonekar (see [5], [4], [12]). Another approach is discussed by Trentelman and Stoorvogel ([6], [7]). We will extend the first approach to nonlinear systems.

This note is organized as follows. In the next section we shall briefly recall some results about the L_2-gain of nonlinear systems. In the third section we shall consider the disturbance attenuation approach used to solve the general (singular) H_∞ problem for linear systems. In the fourth section the singular nonlinear H_∞ state feedback problem for systems of the form (1) will be considered. In the fifth section we will briefly discuss how the theory can be extended to more general systems.

2 The L_2-gain of nonlinear systems

In this section we will first consider the L_2-gain for the system (see [9] and the references quoted in there)

$$\begin{array}{rcl} \dot{x} & = & f(x) + g(x)u \\ y & = & h(x) \end{array} \tag{2}$$

Definition 1 Let γ be a fixed non-negative constant. The system (2) is said to have L_2-gain less than or equal to γ if for all $x \in M$ there exists a constant $K(x)$, $0 \le K(x) < \infty$, with $K(0) = 0$, such that

$$\int_0^T \|y(t)\|^2 \mathrm{d}t \le \gamma^2 \int_0^T \|u(t)\|^2 \mathrm{d}t + K(x)$$

for all $T \ge 0$ and all $u \in L_2(0,T)$ with $y(t)$ denoting the output of (2) resulting from u for initial state $x(0) = x$.
The system has L_2-gain less than γ if there exists some $0 \le \tilde{\gamma} < \gamma$ such that the system (2) has L_2-gain less than or equal to $\tilde{\gamma}$. The L_2-gain is equal to γ if it has L_2-gain less than or equal to γ and not less than γ.

Some fundamental results from dissipative system theory (see e.g. Willems ([11]), Van der Schaft ([9], [10])) will be given without proof in the next theorem

Theorem 2 *The system (2) has L_2-gain less than or equal to γ if and only if there exists a solution $V : M \to \mathbb{R}^+ = \{x \in \mathbb{R} | \mathrm{Re}(x) \ge 0\}$ to the* **integral dissipation inequality**

$$V(x(t_1)) - V(x(t_0)) \le \frac{1}{2} \int_{t_0}^{t_1} (\gamma^2 \|u(t)\|^2 - \|y(t)\|^2) \mathrm{d}t \tag{3}$$

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with $V(0) = 0$ for all $t_1 \geq t_0$ and all $u \in L_2(t_0, t_1)$, where $x(t_1)$ is the state at time t_1 resulting from state $x(t_0)$ at time t_0 and input u. Further, there exists a non-negative C^1-solution to the integral dissipation inequality (3) if and only if there exists a non-negative C^1-solution to the **differential dissipation inequality**

$$\frac{\partial V}{\partial x}(x)f(x) + \frac{\partial V}{\partial x}(x)g(x)u \leq \frac{1}{2}\gamma^2\|u\|^2 - \frac{1}{2}\|y\|^2 \quad (4)$$

with $V(0) = 0$ for all $u \in \mathbb{R}^m$, with $y = h(x)$. And there exists a non-negative C^1-solution to (4) if and only if there exists a non-negative C^1-solution to the **Hamilton-Jacobi inequality**

$$\begin{array}{r}\frac{\partial V}{\partial x}(x)f(x) + \frac{1}{2}\frac{1}{\gamma^2}\frac{\partial V}{\partial x}(x)g(x)g^T(x)\frac{\partial^T V}{\partial x}(x) \\ +\frac{1}{2}h^T(x)h(x) \quad \leq 0\end{array} \quad (5)$$

with $V(0) = 0$ for all $x \in M$.

Also some kind of stability can be concluded from the solvability of the Hamilton-Jacobi equation.

Definition 3 The system (2) is called *zero-state observable* if for any trajectory such that $u(t) \equiv 0, y(t) \equiv 0$ implies $x(t) \equiv 0$.

Theorem 4 *Assume (2) is zero-state observable. Suppose there exists a smooth solution $V \geq 0$ to either (3), (4) or (5). Then $V(x) > 0, x \neq 0$, and the free system $\dot{x} = f(x)$ is locally asymptotically stable. Furthermore, assume that V is proper (i.e., for each $c > 0$ the set $\{x \in M | \; 0 \leq V(x) \leq c\}$ is compact), then $\dot{x} = f(x)$ is globally asymptotically stable.*

In this note we consider the (singular) H_∞ control problem for the nonlinear system (1). We define this problem as follows:

Definition 5 *Singular nonlinear state feedback H_∞ optimal control problem*: Find, if existing, the smallest value $\gamma^* \geq 0$ such that for any $\gamma > \gamma^*$ there exists a state feedback

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} l_1(x) \\ l_2(x) \end{pmatrix} = l(x); \qquad l(0) = 0 \qquad (6)$$

such that the L_2-gain of the closed loop system (1) and (6) from d to z is less than or equal to γ.

We have not included stability in the definition of the H_∞ problem. As above, some kind of stability may be deduced from the solvability of the Hamilton-Jacobi equation as we shall see in section 4.

Furthermore we consider the linearization of (1) around the origin, denoted as

$$\begin{array}{rcl}\dot{\bar{x}} & = & F\bar{x} + G_1\bar{u}_1 + G_2\bar{u}_2 + K\bar{d} \\ \bar{z} & = & \begin{pmatrix} H\bar{x} \\ \bar{u}_1 \end{pmatrix}\end{array} \qquad (7)$$

where $\bar{u} = (\bar{u}_1, \bar{u}_2) \in \mathbb{R}^m$, $\bar{u}_1 \in \mathbb{R}^{m_1}$, $\bar{u}_2 \in \mathbb{R}^{m-m_1}$, $\bar{x} \in \mathbb{R}^n$, $\bar{d} \in \mathbb{R}^q$, $\bar{z} \in \mathbb{R}^p$ and the matrices F, G_1, G_2, K and H defined in the usual way.

We look at the corresponding H_∞ control problem for this system (7). Hence we search for a stabilizing state feedback

$$\bar{u} = \begin{pmatrix} \bar{u}_1 \\ \bar{u}_2 \end{pmatrix} = \begin{pmatrix} L_1\bar{x} \\ L_2\bar{x} \end{pmatrix} = L\bar{x} \qquad (8)$$

such that the H_∞ norm from $\bar{d}$ to $\bar{z}$ is smaller than some value γ. This problem is a special case of the general linear state feedback H_∞ control problem considered by Zhou and Khargonekar ([12]) for the system:

$$\begin{array}{rcl}\dot{\bar{x}} & = & F\bar{x} + G\bar{u} + K\bar{d} \\ \bar{z} & = & C\bar{x} + D_1\bar{u} + D_2\bar{d}\end{array}$$

where in our situation $\bar{z} \in \mathbb{R}^{(p+m_1)}$ and

$$C = \begin{pmatrix} H \\ 0 \end{pmatrix}; \quad D_1 = \begin{pmatrix} 0 & 0 \\ I_{m_1} & 0 \end{pmatrix}; \quad D_2 = 0$$

For the case that (F, G) is stabilizable Petersen, Zhou and Khargonekar ([12], [5]) solved this problem. This solution will be recalled in the next section.

3 Disturbance attenuation

We consider linear systems of the form (7). This system is said to satisfy the *ARE (Algebraic Riccati Equation) with constant* γ if, for arbitrary $Q > 0$, there exists an $\varepsilon > 0$ such that the Riccati equation ($G = (G_1 \; G_2)$)

$$\begin{array}{l}F^TP + PF + \frac{1}{\gamma^2}PKK^TP - PG\begin{pmatrix} I_{m_1} & 0 \\ 0 & 0 \end{pmatrix}G^TP \\ -\frac{1}{\varepsilon}PG\begin{pmatrix} 0 & 0 \\ 0 & I_{m-m_1} \end{pmatrix}G^TP + H^TH + \varepsilon Q = 0\end{array} \qquad (9)$$

has a positive definite solution P.

The following Lemma shows that the existence of a positive definite solution P of ARE (9) does not depend on the choice of Q (see Petersen ([5])).

Lemma 6 *Suppose there exists a positive definite matrix $Q \in \mathbb{R}^{n\times n}$ and a constant $\varepsilon > 0$ such that the algebraic Riccati equation (9) has a positive definite solution. Then given any positive definite $\tilde{Q} \in \mathbb{R}^{n\times n}$ there exists a constant $\varepsilon^* > 0$ such that the ARE (9) with Q replaced by $\tilde{Q}$ has a positive definite solution for all $\varepsilon \in (0, \varepsilon^*]$.*

Now we have the following connection with the H_∞ control problem (see Zhou and Khargonekar ([12])).

Theorem 7 *Consider the system (7). Let $\gamma > 0$. Then there exists a linear feedback of the form (8) such that $F + GL$ is stable and*

$$\left\| \begin{pmatrix} H \\ L_1 \end{pmatrix} (sI - F - GL)^{-1}K \right\|_\infty < \gamma \qquad (10)$$

if and only if for any $Q > 0$ there exists an $\varepsilon > 0$ such that the algebraic Riccati equation (9) has a positive definite solution P.

Moreover if $P > 0$ is a solution of the ARE (9) for some $Q > 0$ and constant $\varepsilon > 0$ then if we choose

$$L = -\begin{pmatrix} I_{m_1} & 0 \\ 0 & \frac{1}{2\varepsilon}I_{m-m_1} \end{pmatrix} G^TP$$

the closed loop system $F + GL$ is stable and (10) holds.

Furthermore if there exists a positive definite solution P of (9), then there also exists a stabilizing solution of (9) (see [4]).

Theorem 8 *Suppose for $Q > 0$ there exists an $\varepsilon^* > 0$ such that (9) has a positive definite solution P_ε for every $\varepsilon \in (0, \varepsilon^*]$. Then for every $\varepsilon \in (0, \varepsilon^*)$ there also exists a stabilizing solution $\tilde{P}_\varepsilon > 0$ for (9), i.e., there exists a solution $\tilde{P}_\varepsilon > 0$ for which also holds that*

$$F + \frac{1}{\gamma^2} K K^T \tilde{P}_\varepsilon - G \begin{pmatrix} I_{m_1} & 0 \\ 0 & \frac{1}{\varepsilon} I_{m-m_1} \end{pmatrix} G^T \tilde{P}_\varepsilon$$

is asymptotically stable.

4 Singular nonlinear state feedback H_∞ optimal control

Now consider the singular state feedback H_∞ optimal control problem for an affine nonlinear systems of the form (1) for which we seek a nonlinear static feedback of the form (6) such that the closed loop system (1), (6) has L_2-gain less than (or equal to) γ.

As mentioned before, Definition 5 of the state feedback H_∞ optimal control problem is stated without any considerations about asymptotic stability. Of course we want the closed loop system to be asymptotically stable but we consider that point separately.

We start with a theorem, which is a simple extension of a result obtained in [9] where the L_2-gain from disturbance d to output y and the complete input u was considered.

Theorem 9 *Consider the nonlinear system (1). Let $\gamma > 0$. Suppose there exists a constant $\varepsilon > 0$ such that there exists a non-negative C^r-solution V to the $(k \geq r \geq 1)$* **Hamilton-Jacobi inequality***:*

$$\begin{aligned} &\tfrac{\partial V}{\partial x}(x) f(x) + \tfrac{1}{2}\tfrac{\partial V}{\partial x}(x)\left[\tfrac{1}{\gamma^2} k(x) k^T(x) - g_1(x) g_1^T(x)\right. \\ &\qquad \left. - \tfrac{1}{\varepsilon} g_2(x) g_2^T(x)\right] \tfrac{\partial^T V}{\partial x}(x) + \tfrac{1}{2} h^T(x) h(x) \leq 0 \end{aligned} \tag{11}$$

with $V(0) = 0$ then the closed loop system for feedback

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = - \begin{pmatrix} g_1^T(x) \\ \frac{1}{2\varepsilon} g_2^T(x) \end{pmatrix} \frac{\partial^T V}{\partial x}(x) \tag{12}$$

has L_2-gain (from d to z) less than or equal to γ.

Proof Using Theorem 2 on the system

$$\begin{aligned} \dot{x} &= f(x) - g_1(x) g_1^T(x) \tfrac{\partial^T V}{\partial x}(x) - \tfrac{1}{2\varepsilon} g_2(x) g_2^T(x) \tfrac{\partial^T V}{\partial x} \\ &\quad + k(x) d \\ z &= \begin{pmatrix} h(x) \\ -g_1^T(x) \frac{\partial^T V}{\partial x}(x) \end{pmatrix} \end{aligned} \tag{13}$$

leads to the result. ∎

From this theorem we can easily construct the following condition under which the system has L_2-gain less than γ.

Corollary 10 *Consider the system (1) and let $\gamma > 0$. Suppose there exists constants $\varepsilon, \mu > 0$ such that there exists a non-negative C^r-solution V $(k \geq r \geq 1)$ to the Hamilton-Jacobi inequality:*

$$\begin{aligned} &\tfrac{\partial V}{\partial x}(x) f(x) + \tfrac{1}{2}\tfrac{\partial V}{\partial x}(x)\left[(\tfrac{1}{\gamma^2} + \mu) k(x) k^T(x)\right. \\ &\quad \left. - g_1(x) g_1^T(x) - \tfrac{1}{\varepsilon} g_2(x) g_2^T(x)\right] \tfrac{\partial^T V}{\partial x}(x) \\ &\quad + \tfrac{1}{2} h^T(x) h(x) \;\leq\; 0 \end{aligned} \tag{14}$$

with $V(0) = 0$ then the closed loop system for the feedback (12) has L_2-gain less than γ.

Proof From (14) it follows that there exists an constant $0 \leq \tilde{\gamma} < \gamma$ such that (11) is satisfied with γ replaced by $\tilde{\gamma}$. Hence by theorem 9 the closed loop system has L_2-gain less than γ (less than or equal to $\tilde{\gamma}$). ∎

The (partial) converse of Corollary 10 is a little more involved. Let us assume that there exists a static feedback (6) which leads to a closed loop system which has L_2-gain less than γ. From Definition 1 we know that there exists a constant $\eta > 0$ such that for all $x \in M$ there exists a constant $K(x)$, $0 \leq K(x) < \infty$ with $K(0) = 0$, such that

$$\begin{aligned} &\textstyle\int_0^T \left(\|l_1(x(t))\|^2 + \|h(x(t))\|^2\right) \mathrm{d}t \\ &\qquad \leq (\gamma^2 - \eta) \textstyle\int_0^T \|d(t)\|^2 \mathrm{d}t + K(x) \end{aligned}$$

holds for all $d \in L_2[0, T]$ and all $T > 0$.

Now we assume that the L_2-gain from disturbance d to the feedback $l_2(x)$ is finite, i.e., there exists a constant $N > 0$ such that for every $x \in M$ there exists a constant $\tilde{K}(x)$, $0 \leq \tilde{K}(x) < \infty$, with $\tilde{K}(0) = 0$, such that

$$\int_0^T \|l_2(x(t))\|^2 \mathrm{d}t \leq N \int_0^T \|d(t)\|^2 \mathrm{d}t + \tilde{K}(x)$$

for all $d \in L_2[0, T]$ and all $T \geq 0$. Then by choosing ε small enough $(0 < \varepsilon < \eta/N)$ the L_2-gain from d to y, l_1 and $\sqrt{\varepsilon} l_2$ will be less than γ. Hence there exists a solution $V \geq 0$ to the following integral dissipation inequality:

$$\begin{aligned} V(x(t_1)) - V(x(t_0)) &\leq \tfrac{1}{2} \textstyle\int_{t_0}^{t_1} \left((\gamma^2 - \mu)\|d(t)\|^2\right. \\ &\quad \left. - \|y(t)\|^2 - \|l_1(x(t))\|^2 - \varepsilon \|l_2(x(t))\|^2\right) \mathrm{d}t \end{aligned} \tag{15}$$

with $V(0) = 0$ for all $t_1 \geq t_0$ and all $d \in L_2(t_0, t_1)$ and for some $\mu > 0$.

At this point we assume that there exists a C^1-solution $V \geq 0$ to the above integral dissipation inequality. Then there exists a C^1-solution $V \geq 0$ to the Hamilton-Jacobi inequality:

$$\begin{aligned} &\tfrac{\partial V}{\partial x}(x) \left(f(x) + g(x) l(x)\right) \\ &+ \tfrac{1}{2}(\tfrac{1}{\gamma^2} + \mu) \tfrac{\partial V}{\partial x}(x) k(x) k^T(x) \tfrac{\partial^T V}{\partial x} + \tfrac{1}{2} h^T(x) h(x) \\ &\qquad + \tfrac{1}{2} l_1^T(x) l_1(x) + \tfrac{1}{2} \varepsilon l_2^T(x) l_2(x) \;\leq\; 0 \end{aligned}$$

with $V(0) = 0$ for all x and for some $\mu > 0$. Now by the following completion of the squares argument

$$\begin{aligned} &\tfrac{\partial V}{\partial x}\left(f - g_1 g_1^T \tfrac{\partial^T V}{\partial x} - \tfrac{1}{\varepsilon} g_2 g_2^T \tfrac{\partial^T V}{\partial x}\right) \\ &= \tfrac{\partial V}{\partial x}\left(f + g_1 l_1 + g_2 l_2\right) + \tfrac{\partial V}{\partial x} g_1 \left(-g_1^T \tfrac{\partial^T V}{\partial x} - l_1\right) \\ &\quad + \tfrac{\partial V}{\partial x} g_2 \left(-\tfrac{1}{\varepsilon} g_2^T \tfrac{\partial^T V}{\partial x} - l_2\right) \\ &\leq -\tfrac{1}{2}(\tfrac{1}{\gamma^2} + \mu) \tfrac{\partial V}{\partial x} k k^T \tfrac{\partial^T V}{\partial x} - \tfrac{1}{2} h^T h - \tfrac{1}{2}\left\|l_1 + g_1^T \tfrac{\partial^T V}{\partial x}\right\|^2 \\ &\quad - \tfrac{1}{2} \tfrac{\partial V}{\partial x} g_1 g_1^T \tfrac{\partial^T V}{\partial x} - \tfrac{1}{2} \varepsilon \left\|l_2 + \tfrac{1}{\varepsilon} g_2^T \tfrac{\partial^T V}{\partial x}\right\|^2 \\ &\quad - \tfrac{1}{2\varepsilon} \tfrac{\partial V}{\partial x} g_2 g_2^T \tfrac{\partial^T V}{\partial x} \end{aligned}$$

it can be concluded that V is also a solution of equation (14) and hence by corollary 10 it follows that also the feedback (12) leads to a closed loop system which has L_2-gain less than γ. Summarizing this result:

Theorem 11 *Consider the system (1), and let $\gamma > 0$. Suppose there exists a smooth feedback of the form (6) such*

that the L_2-gain of the closed loop system is less than γ and assume that the L_2-gain from disturbance d to the feedback $l_2(x)$ is finite and also make the assumption that one of the solutions of (15) is C^1.
Then V is also a solution of equation (14) for some $\mu > 0$, and hence the closed loop system for the feedback (12) has L_2-gain less than γ.

Remark 12 Corollary 10 together with Theorem 11 give under an assumption about the L_2 gain of the feedback $l_2(x)$ a necessary and sufficient condition for the existence of a feedback for the system (1) which achieves an L_2 gain less than a certain bound γ. The same result is for linear systems in fact already proved by Petersen ([5]) for stabilizing feedbacks.

Until now we have not considered the stability of the closed loop system. But the following theorem can be easily obtained from Theorem 4.

Theorem 13 *Suppose there exists a solution $V \geq 0$ to (11). Assume the system (13) is zero-state observable. Then $V(x) > 0$ for $x \neq 0$ and the closed loop system (1), (12) (with $d(t) \equiv 0$) is locally asymptotically stable.*
Assume additionally that V is proper, then the closed loop system is globally asymptotically stable.

Now we consider the linearization (7) of the nonlinear system (1) around the origin. Straightforwardly from Van der Schaft ([9]) the next stability result can be obtained.

Proposition 14 *Suppose the L_2-gain of (1), (6) is less than γ, and assume $F + GL$ with $L = \frac{\partial l}{\partial x}(0)$ is asymptotically stable, then there exists a neighborhood W of 0 and a smooth function $V \geq 0$ on W satisfying (11).*
Alternatively, assume $f + gl$ is globally asymptotically stable. Define the Hamiltonian

$$\begin{aligned} H_\gamma(x,p) &= p^T[f(x) + g(x)l(x)] + \tfrac{1}{2}\tfrac{1}{\gamma^2}p^T k(x)k^T(x)p \\ &\quad + \tfrac{1}{2}h^T(x)h(x) + \tfrac{1}{2}l_1^T(x)l_1(x) \end{aligned}$$

and suppose X_{H_γ} is hyperbolic, and its stable invariant manifold is diffeomorphic to M under the canonical projection $\pi : T^*M \to M$. Then there exists a global solution $V \geq 0$ to (11).

We search for a static feedback of the form (8) such that the closed loop linear system (7), (8) has L_2-gain less than (or equal to) γ.
This is always possible if the nonlinear closed loop system (1), (6) has L_2-gain less than (or equal to) γ.

Proposition 15 *Let $\gamma > 0$. Suppose there exists a smooth feedback $u = l(x)$, $l(0) = 0$, for (1) such that the L_2-gain of the nonlinear closed loop system (1), (6) is less than (or equal to) γ. Then the linear feedback $\bar{u} = L\bar{x}$, with $L = \frac{\partial l}{\partial x}(0)$, for (7) results in the linear closed loop system*

$$\begin{aligned} \dot{\bar{x}} &= (F + GL)\bar{x} + K\bar{d} \\ \bar{z} &= \begin{pmatrix} H \\ L_1 \end{pmatrix}\bar{x} \end{aligned} \tag{16}$$

with also has L_2-gain less than (or equal to) γ.

Proof The linearization of (1), (6) is equal to (16). Then the result follows from [9]. ∎

Then the converse result. Given a feedback $\bar{u} = L\bar{x}$ which solves the H_∞ problem for the linearized system then what can we say about the H_∞ problem for the nonlinear system.

Theorem 16 *Consider the linearized system (7). Let $\gamma > 0$. Suppose there exists a feedback $\bar{u} = L\bar{x}$ such that the L_2-gain of the closed loop system (from $\bar{d}$ to $\bar{z}$) is* less than *γ and the closed loop system is asymptotically stable.*
Then there exists a neighborhood W of 0 and a smooth function $V \geq 0$ defined on W such that V is a solution of the Hamilton-Jacobi inequality (11). Furthermore, the feedback (12) for (1) has the property that the closed loop system has locally L_2-gain less than *γ, in the sense that for all $x \in W$ there exists a constant $K(x)$, $0 \leq K(x) < \infty$, with $K(0) = 0$, such that*

$$\int_0^T \left(\|y(t)\|^2 + \|u_1(t)\|^2\right) \mathrm{d}t \leq \gamma^2 \int_0^T \|d(t)\|^2 \mathrm{d}t + K(x)$$

for all $T \geq 0$ and all $d \in L_2(0,T)$ such that the state space trajectories $x(t)$ starting from $x(0) = x$ do not leave W (i.e., the state feedback H_∞-control problem for γ is solved on W).

Proof The L_2-gain of the closed loop linearized system is less than γ then there exists a $\tilde{\gamma} < \gamma$ such that the L_2-gain is less than or equal to $\tilde{\gamma}$. But then for every $\hat{\gamma}$ for which $\tilde{\gamma} < \hat{\gamma} < \gamma$ it holds that the L_2-gain is less than $\hat{\gamma}$. Hence there exists by theorem 7 and 8 for every $Q > 0$ an $\varepsilon^* > 0$ and a stabilizing solution $P_\varepsilon > 0$ of the ARE

$$\begin{aligned} F^TP + PF + (\tfrac{1}{\gamma^2} + \eta)PKK^TP - PG\begin{pmatrix} I_{m_1} & 0 \\ 0 & 0 \end{pmatrix}G^TP \\ -\tfrac{1}{\varepsilon}PG\begin{pmatrix} 0 & 0 \\ 0 & I_{m-m_1} \end{pmatrix}G^TP + H^TH + \varepsilon Q = 0 \end{aligned}$$

for some $\eta > 0$ and $\varepsilon \in (0, \varepsilon^*]$. Now take the Hamiltonian

$$\begin{aligned} H_\gamma(x,p) &:= p^Tf(x) + \tfrac{1}{2}p^T\Big((\tfrac{1}{\gamma^2} + \eta)k(x)k^T(x) \\ &\quad - g_1(x)g_1^T(x) - \tfrac{1}{\varepsilon}g_2(x)g_2^T(x)\Big)p \\ &\quad + \tfrac{1}{2}h^T(x)h(x) + \varepsilon q(x) \end{aligned}$$

where q is an arbitrary function $q : M \to \mathbb{R}^+$ which satisfies that

$$q(0) = 0, \quad \frac{\partial q}{\partial x}(0) = 0, \quad \frac{\partial^2 q}{\partial x^2}(0) = Q > 0$$

Then the linearization of X_{H_γ} at $(0,0)$ is given by the Hamiltonian matrix $DX_{H_\gamma}(0,0)$ defined as

$$\begin{pmatrix} F & (\tfrac{1}{\gamma^2} + \eta)KK^T - G\begin{pmatrix} I_{m_1} & 0 \\ 0 & \tfrac{1}{\varepsilon}I_{m-m_1} \end{pmatrix}G^T \\ -H^TH - \varepsilon Q & -F^T \end{pmatrix}$$

Then $P_\varepsilon = P_\varepsilon^T$ is a solution of (9) if and only if

$$\begin{aligned} DX_{H_\gamma}(0,0)\begin{bmatrix} I \\ P_\varepsilon \end{bmatrix} = \\ \begin{bmatrix} I \\ P_\varepsilon \end{bmatrix}\left(F + (\tfrac{1}{\gamma^2} + \eta)KK^TP_\varepsilon - G\begin{pmatrix} I_{m_1} & 0 \\ 0 & \tfrac{1}{\varepsilon}I_{m-m_1} \end{pmatrix}G^TP_\varepsilon\right) \end{aligned}$$

and thus

$$\text{span}\begin{bmatrix} I \\ P_\varepsilon \end{bmatrix} = \text{stable eigenspace of } DX_{H_\gamma}(0,0)$$

for some ε. Thus the Hamiltonian matrix $DX_{H_\gamma}(0,0)$ does not have imaginary eigenvalues.
Then by Proposition A4 and A7 from [9] the stable invariant manifold N^- of X_{H_γ} through $(0,0)$ is n-dimensional and is tangent at $(0,0)$ to span $\begin{bmatrix} I \\ P_\varepsilon \end{bmatrix}$ (for $\varepsilon \in (0,\varepsilon^*)$).
Furthermore locally around 0 the manifold N^- is given as

$$N^- = \left\{ \left(x, p = \frac{\partial^T V}{\partial x}(x) \right) \middle| \, x \text{ around } 0 \right\}$$

where V is a (local) solution of the Hamilton-Jacobi equation

$$\begin{array}{l} \frac{\partial V}{\partial x} f + \frac{1}{2}\frac{\partial V}{\partial x}\Big[(\frac{1}{\gamma^2}+\eta)kk^T \\ -g\begin{pmatrix} I_{m_1} & 0 \\ 0 & \frac{1}{\varepsilon}I_{m-m_1} \end{pmatrix} g^T \Big] \frac{\partial^T V}{\partial x} + \frac{1}{2}h^T h + \varepsilon q = 0 \end{array}$$

with $\frac{\partial^2 V}{\partial x^2}(0) = P$. From Proposition 3 from [9] $V \geq 0$ because $F - G\begin{pmatrix} I_{m_1} & 0 \\ 0 & \frac{1}{\varepsilon}I_{m-m_1} \end{pmatrix} G^T P_\varepsilon$ is asymptotically stable then the final result follows from Corollary 10. ∎

Then we will summarize some of the above results in the following theorem

Theorem 17 *Consider the nonlinear system (1) and its linearization (7). Then the following statements are equivalent:*

(a) There exists a linear feedback of the form (8) such that the L_2-gain of the closed loop system (7), (8) is less than γ and $F + GL$ is stable.

(b) There exists a positive definite solution P to the Algebraic Riccati Equation

$$\begin{array}{r} F^TP + PF + \frac{1}{\gamma^2}PKK^TP - PG_1G_1^TP \\ -\frac{1}{\varepsilon}PG_2G_2^TP + H^TH + \varepsilon Q = 0 \end{array}$$

for all $Q > 0$ and for some $\varepsilon > 0$ such that also

$$F + \frac{1}{\gamma^2}KK^TP - G_1G_1^TP - \frac{1}{\varepsilon}G_2G_2^TP$$

is asymptotically stable.

(c) There exists a neighborhood $W \subset M$ of 0, and a nonlinear feedback $u = l(x)$ as in (6) defined on W, such that $F + GL$, with $L = \frac{\partial l}{\partial x}(0)$, is asymptotically stable and the closed loop system (1), (6) has locally L_2-gain less than γ on W.

Remark 18 In the regular case (see [9]), $m_1 = m$, there was the extra assumption that (H, F) must be detectable. Now $H^TH + \varepsilon Q > 0$ and hence there exists a nonsingular matrix $\tilde{H}$ (Cholesky decomposition) such that

$$\tilde{H}^T\tilde{H} = H^TH + \varepsilon Q$$

Because of the nonsingularity of $\tilde{H}$ $(\tilde{H}, F)$ is always detectable.

5 Extensions

Parallel to the theory for linear systems ([5], [4], [12]) we can easily extend most of the results obtained above to systems of the form

$$\begin{array}{rcl} \dot{x} & = & f(x) + g(x)u + k(x)d \\ z & = & h(x) + m(x)u \end{array} \tag{17}$$

where $u \in \mathbb{R}^m$, $d \in \mathbb{R}^q$, $z \in \mathbb{R}^p$ and $x \in M$. Furthermore $x = 0$ is an equilibrium and $f(0) = 0$, $h(0) = 0$.
Suppose locally $\text{rank}(m(x)) = m_1 \leq m$. Then it is always possible to find locally an $(p \times m_1)$-matrix $n(x)$ and a constant $(m_1 \times m)$-matrix Π such that $\text{rank}(\Pi) = \text{rank}(n(x)) = m_1$ and

$$m(x) = n(x)\Pi$$

Let $\Phi \in \mathbb{R}^{(m-m_1)\times m}$ be such that

$$\Phi\Pi^T = 0$$

and than define

$$\sigma(x) := \Pi^T(\Pi\Pi^T)^{-1}\{n^T(x)n(x)\}^{-1}(\Pi\Pi^T)^{-1}\Pi$$

Then Theorem 9 can be extended to

Theorem 19 *Consider the nonlinear system (17). Let $\gamma > 0$. Suppose there exists a constant $\varepsilon > 0$ such that there exists a non-negative C^r-solution V to the Hamilton-Jacobi inequality $(k \geq r \geq 1)$.*

$$\begin{array}{l} \frac{\partial V}{\partial x}(x)(f(x) - g(x)\sigma(x)m^T(x)h(x)) \\ +\frac{1}{2}\frac{\partial V}{\partial x}(x)\left[\frac{1}{\gamma^2}k(x)k^T(x) - g(x)\sigma(x)g^T(x) \right. \\ \qquad \left. -\frac{1}{\varepsilon}g(x)\Phi^T\Phi g^T(x)\right] \frac{\partial^T V}{\partial x}(x) \\ +\frac{1}{2}h^T(x)(I_p - m(x)\sigma(x)m^T(x))h(x) \;\leq\; 0 \end{array}$$

with $V(0) = 0$ then the closed loop system for the feedback

$$\begin{array}{rcl} u(x) & = & -\left(\frac{1}{2\varepsilon}\Phi^T\Phi + \sigma(x)\right)g^T(x)\frac{\partial^T V}{\partial x}(x) \\ & & -\sigma(x)m^T(x)h(x) \end{array} \tag{18}$$

has (locally) L_2-gain less than or equal to γ.

Proof Again as in the proof of Theorem 9 apply Theorem 2 to the closed loop system (17) and (18). ∎

And then also Corollary 10 has the obvious generalisation

Corollary 20 *Consider the nonlinear system (17). Let $\gamma > 0$. Suppose there exists a constants $\varepsilon, \mu > 0$ such that there exists a non-negative C^r-solution V to the Hamilton-Jacobi inequality $(k \geq r \geq 1)$.*

$$\begin{array}{l} \frac{\partial V}{\partial x}(x)(f(x) - g(x)\sigma(x)m^T(x)h(x)) \\ +\frac{1}{2}\frac{\partial V}{\partial x}(x)\left[(\frac{1}{\gamma^2}+\mu)k(x)k^T(x) - g(x)\sigma(x)g^T(x) \right. \\ \qquad \left. -\frac{1}{\varepsilon}g(x)\Phi^T\Phi g^T(x)\right] \frac{\partial^T V}{\partial x}(x) \\ +\frac{1}{2}h^T(x)(I_p - m(x)\sigma(x)m^T(x))h(x) \;\leq\; 0 \end{array}$$

with $V(0) = 0$ then the closed loop system for the feedback (18) has (locally) L_2-gain less than γ.

In the same way as in the Theorems 16 and 17 we can consider the linearization of (17) around the equilibrium $x = 0$:

$$\begin{aligned} \dot{\bar{x}} &= F\bar{x} + G\bar{u} + K\bar{d} \\ \bar{z} &= H\bar{x} + M\bar{u} \end{aligned} \tag{19}$$

where $\bar{u} \in \mathbb{R}^m$, $\bar{x} \in \mathbb{R}^n$, $\bar{d} \in \mathbb{R}^q$, $\bar{z} \in \mathbb{R}^p$ and the matrices F, G, K, H and M defined in the usual way.
Now define the linear equivalents of $n(x)$ and $\sigma(x)$ as

$$U = u(0); \quad \Sigma = \sigma(0)$$

Then Theorem 16 and 17 can be extended straightforwardly to

Theorem 21 *Consider the linearized system (19). Let $\gamma > 0$. Suppose there exists a feedback $\bar{u} = L\bar{x}$ such that the L_2-gain of the closed loop system (from $\bar{d}$ to $\bar{z}$) is* less than *γ and the closed loop system is asymptotically stable. Then there exists a neighborhood W of x_0 and a smooth function $V \geq 0$ defined on W such that V is a solution of the Hamilton-Jacobi inequality (11). Furthermore, the feedback (18) for (17) has the property that the closed loop system has locally L_2-gain* less than γ.

Theorem 22 *Consider the nonlinear system (17) and its linearization (19). Then the following statements are equivalent:*

(a) There exists a linear feedback of the form (8) such that the L_2-gain of the closed loop system (19), (8) is less than γ and $F + GL$ is stable.

(b) There exists a positive definite solution P to the Algebraic Riccati Equation

$$\begin{aligned} &(F - G\Sigma M^T H)^T P + P(F - G\Sigma M^T H) \\ &+\tfrac{1}{\gamma^2} PKK^T P - PG\Sigma G^T P - \tfrac{1}{\varepsilon} PG\Phi^T \Phi G^T P \\ &\qquad + H^T(I - M\Sigma M^T)H + \varepsilon Q \quad = 0 \end{aligned}$$

for all $Q > 0$ and for some $\varepsilon > 0$ which also satisfies that

$$F + \frac{1}{\gamma^2} KK^T P - G\Sigma(G^T P + M^T H) - \frac{1}{\varepsilon} G\Phi^T \Phi G^T P$$

is asymptotically stable.

(c) There exists a neighborhood $W \subset M$ of 0, and a nonlinear feedback $u = l(x)$ defined on W, such that $F + GL$, with $L = \frac{\partial l}{\partial x}(0)$, is asymptotically stable and the closed loop system of this nonlinear feedback and the system (17) has locally L_2-gain less than γ on W.

6 Conclusions

We have extended both the theory about the regular nonlinear state feedback H_∞ problem ([8], [9], [3] and [1]) to the more general setting of singular nonlinear systems and the theory about disturbance attenuation from Khargonekar, Petersen and Zhou ([4], [5] and [12]) for singular linear systems to nonlinear systems.

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